

Recall: $F(\underline{x}) \in \mathbb{Z}[x_0, \dots, x_s]$, degree d

$$G(\alpha) = \sum_{\underline{x} \in \mathbb{B} \cap \mathbb{Z}^{s+1}} e(\alpha F(\underline{x}))$$

$$\begin{aligned} \# \{ \underline{x} \in \mathbb{B} \cap \mathbb{Z}^{s+1} : F(\underline{x}) = 0 \} \\ = \int_0^1 G(\alpha) d\alpha \end{aligned}$$

$\psi(B) \nearrow \infty$ slowly

$$\mathcal{M} = \bigcup_{\substack{0 \leq a \leq q \leq \psi(B) \\ (a, q) = 1}} \mathcal{M}(q, a)$$

Major Arcs

$$\mathcal{M}(q, a) = \{ \alpha \in [0, 1) : |\alpha - \frac{a}{q}| \leq \psi(B) B^{-d} \}$$

$$\mathcal{m} = [0, 1) \setminus \mathcal{M}$$

Minor Arcs

$$\int_0^1 G(\alpha) d\alpha = \int_{\mathcal{M}} G(\alpha) d\alpha + \int_{\mathcal{m}} G(\alpha) d\alpha$$

$$\prod_p \left(\sum_{\substack{\alpha \in \mathbb{Z}^{s+1} \\ \alpha \equiv 0 \pmod{p}} } e(\alpha F(\underline{x})) \right) B^{s+1-d} \sim \text{"Product of local densities"}$$

$$\text{small } o(B^{s+1-d})$$

IX. OTHER FIELDS.

Circle method is quite general approach.

Details (usually minor arc treatment) can be sensitive to characteristics of setting, but relatively robust.

Example 1. (Number fields)

Let K be an algebraic extension of $\mathbb{Q}$, say $[K:\mathbb{Q}] = n$.

Let $K^{(l)}$ ($1 \leq l \leq r_1$) be the real conjugates, $K^{(m)}$, $K^{(m+r_2)}$ ($r_1+1 \leq m \leq r_1+r_2$) the complex conjugates ($n = r_1 + 2r_2$).

Also, when $y \in K$ let $y^{(i)}$ ($1 \leq i \leq n$) denote the conjugates of y with $y^{(i)} \in K^{(i)}$ ($1 \leq i \leq n$).

$\mathcal{O}_K$: = ring of integers of K

When $\gamma_j \in K$, $\alpha_j \in \mathbb{R}$ ($1 \leq j \leq n$),

put

$$\gamma_{\underline{\alpha}} = \sum_{j=1}^n \gamma_j \alpha_j$$

$$\gamma_{\underline{\alpha}}^{(i)} := \sum_{j=1}^n \gamma_j^{(i)} \alpha_j \quad (1 \leq i \leq n)$$

Let $\omega_1, \dots, \omega_n$ denote an integral basis of $\mathcal{O}_K$,

let δ be the different (ground ideal)

$\rho_1, \dots, \rho_n$ basis of δ^{-1}

- satisfies $\sum_{i=1}^n \rho_i^{(1)} \omega_j^{(1)} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$

Detector function:

$$E(\gamma_{\underline{\alpha}}) = e \left(\sum_{l=1}^n \gamma_{\underline{\alpha}}^{(l)} \right)$$

$$e(z) = e^{2\pi i z}$$



$$\int_{[0,1]^n} E(\lambda \rho_{\underline{\alpha}}) d\underline{\alpha} = \begin{cases} 1, & \lambda = 0 \\ 0, & \lambda \in \mathcal{O}_K^* \end{cases}$$

Now consider a bounded region $B \subseteq K^{s+1}$
(e.g. relative to coordinate basis), and

$$F(\underline{x}) \in \mathcal{O}_K[x_0, \dots, x_s].$$

Define

$$G(\underline{\alpha}) = \sum_{\underline{\lambda} \in B \cap \mathcal{O}_K^{s+1}} E(F(\underline{\lambda}) \rho_{\underline{\alpha}}).$$

Then

$$\# \{ \underline{\lambda} \in B \cap \mathcal{O}_K^{s+1} : F(\underline{\lambda}) = 0 \}$$

$$= \int_{[0,1]^n} G(\underline{\alpha}) d\underline{\alpha}$$

Now must define major arcs.

Given $\gamma \in K$, can associate integral ideals $\alpha, \mathfrak{b}$ (uniquely) via

$$\gamma \delta = \mathfrak{b}/\alpha, \quad (\alpha, \mathfrak{b}) = 1.$$

Now define major arcs $\mathcal{M}$ to be union of small neighbourhoods of points $\underline{\alpha} \in [0,1)^n$ for which $\rho_{\underline{\alpha}}$ is associated to $\mathfrak{b}/\alpha$ with $(\alpha, \mathfrak{b}) = 1$ and $N(\alpha)$ suitably small

By (essentially) formal calculation,

$$\int_{\mathcal{M}} G(\underline{\alpha}) d\underline{\alpha} \sim \text{"expected product of local densities."}$$

Seek conclusions where number of variables needed does not depend on $[K:\mathbb{Q}]$

Example 2. (Function fields)

Let $\mathbb{F}_q[t]$ be polynomial ring over $\mathbb{F}_q$,
finite field of characteristic p

$\mathbb{F}_q(t)$ - field of fractions

$$\mathbb{F}_q(t)_\infty := \mathbb{F}_q((1/t)) = \left\{ \alpha = \sum_{i=-\infty}^n a_i t^i : a_i \in \mathbb{F}_q \right\}$$

When $\alpha \in \mathbb{F}_q(t)_\infty$, define

$$\text{ord } \alpha = n \quad \text{where } n = \max_i \{a_i \neq 0\}$$

(ord 0 = $-\infty$)

$$\Pi := \{ \alpha \in \mathbb{F}_q(t)_\infty : \text{ord } \alpha < 0 \}$$

(compact additive subgroup of
 $\mathbb{F}_q(t)_\infty$)

Note: For each $\alpha \in \mathbb{F}_q(t)_\infty$, can write
 $\alpha = [\alpha] + \|\alpha\|$,
with $[\alpha] \in \mathbb{F}_q[t]$ & $\|\alpha\| \in \Pi$.

Given Haar measure $d\alpha$ on $\mathbb{F}_q(t)_\infty$,
can normalize with $\int_{\Pi} d\alpha = 1$.

There is a non-trivial character

$$e_q : \mathbb{F}_q \rightarrow \mathbb{C}$$

defined by $e_q(a) = e\left(\frac{\text{tr } a}{p}\right) \quad (a \in \mathbb{F}_q)$

$\downarrow$

$$e : \mathbb{F}_q(t)_\infty \rightarrow \mathbb{C}$$

$$\alpha = \sum_{i=-\infty}^n a_i t^i \longmapsto e_q(\text{res } \alpha) = e_q(a_{-1})$$

Fact: $\int_{\Pi} e(\alpha f) d\alpha = \begin{cases} 1, & f=0 \\ 0, & \begin{cases} f \in \mathbb{F}_q[t] \\ f \neq 0 \end{cases} \end{cases}$

Now consider $F(\underline{x}) \in \mathbb{F}_q[t][x_0, \dots, x_s]$.
and define

$$G(\underline{\alpha}) = \sum_{\substack{\underline{x} \in \mathbb{F}_q[t]^{s+1} \\ \deg x_i \leq B}} e(\underline{\alpha} F(\underline{x})).$$

Then

$$\# \left\{ \underline{x} \in \mathbb{F}_q[t]^{s+1}, \deg x_i \leq B \ (0 \leq i \leq s); \right. \\ \left. F(\underline{x}) = 0 \right\}$$

$$= \int_{\Pi} G(\underline{\alpha}) d\underline{\alpha}$$

Need to define major arcs:

Given $g, a \in \mathbb{F}_q[t]$, with g monic,
 $(a, g) = 1$ and $\text{ord } a < \text{ord } g$, define

$$M(g, a) = \{ \alpha \in \mathbb{T} : \text{ord}(g\alpha - a) < \psi(B) - dB \}$$

where $d = \deg F(x)$ and $\psi(B) \nearrow \infty$
 slowly.

Major arcs $\mathcal{M} := \bigcup_{\substack{a, g \in \mathbb{F}_q[t] \\ g \text{ monic}, (a, g) = 1 \\ \text{ord } a < \text{ord } g \\ \text{ord } g \leq \psi(B)}} M(g, a)$

Almost formal:

$$\int_{\mathcal{M}} G(\alpha) d\alpha \sim \text{"expected product of local densities"}$$

$$\ll \binom{B}{q}^{s+1-d}$$

Hard results here are ones
independent of characteristic

Not much in literature, but fairly
clear analogues of classical ($\mathbb{Q}$)
results whenever $\text{char}(\mathbb{F}_q) > d$
(degree of polynomials involved).

Theorem: (Yu-Ru Liu & W., 2006)

Independent of characteristic of $\mathbb{F}_q[t]$,
one has a lower bound of expected
size for number of solutions of

$$a_0 x_0^d + \dots + a_s x_s^d = 0 \quad (a_i \in \mathbb{F}_q[t] \text{ fixed})$$

with $\underline{x} \in \mathbb{F}_q[t]^{s+1}$, $\deg x_i \leq B$, whenever

$$s \geq \frac{4}{3} d \log d + O(d \log \log d).$$

X. MINOR ARC ESTIMATES - WEYL.

Weyl, 1916, Hardy & Littlewood, 1920's...

Illustrate first with simplest example

$$f(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^k).$$

$$e(z) = e^{2\pi i z}$$

Since (right now) we know nothing except
When α is close to a rational with
small denominator, let's try to simplify.

$$\begin{aligned} |f(\alpha)|^2 &= f(\alpha) f(-\alpha) \\ &= \sum_{1 \leq x \leq P} \sum_{1 \leq y \leq P} e(\alpha(y^k - x^k)) \\ &= \sum_{1 \leq x \leq P} \sum_{\substack{1-x \leq h \leq P-x \\ y = x+h}} e(\alpha((x+h)^k - x^k)) \end{aligned}$$

$$|f(\alpha)|^2 = \sum_{|h| \leq P} \sum_{x \in I(h)} e(\Delta(x^k; h)\alpha)$$

where $I(h) = [1, P] \cap [1-h, P-h]$

and
$$\begin{aligned} \Delta(x^k; h) &= (x+h)^k - x^k \\ &= h(kx^{k-1} + \binom{k}{2}hx^{k-2} + \dots) \\ &= h p_{k-1}(x; h) \end{aligned}$$

So we've "simplified" exponential sum
over degree k polynomial

↓
degree $k-1$ polynomial (in x)

Now iterate:

$$\begin{aligned} |f(\alpha)|^4 &= \left| \sum_{|h| \leq P} \sum_{x \in I(h)} e(\Delta(x^k; h)\alpha) \right|^2 \\ &\leq (2P+1) \sum_{|h| \leq P} \left| \sum_{x \in I(h)} e(\Delta(x^k; h)\alpha) \right|^2 \end{aligned}$$

$$|f(\alpha)|^4 \leq (2P+1) \sum_{|h_1| < P} \sum_{|h_2| < P} \sum_{x \in I(h_1, h_2)} e(\Delta_2(x^k; h_1, h_2)\alpha)$$

where $I(h_1, h_2) = I(h_1) \cap \{x \in [1, P]: x+h_2 \in I(h_1)\}$

$$\begin{aligned} \Delta_2(x^k; h_1, h_2) &= \Delta((x+h_2)^k; h_1) - \Delta(x^k; h_1) \\ &= (x+h_1+h_2)^k - (x+h_1)^k \\ &\quad - (x+h_2)^k + x^k \end{aligned}$$

$$= k(k-1) h_1 h_2 (x^{k-2} + \dots)$$

$$|f(\alpha)|^{2^j} \leq (2P+1)^{2^j - j - 1} \sum_{\substack{|h_1| < P \\ \vdots \\ |h_j| < P}} \sum_{x \in I_j(\underline{h})} e(\Delta_j(x^k; \underline{h})\alpha)$$

with

$$\Delta_j(x^k; \underline{h}) = k(k-1)\dots(k-j+1)h_1\dots h_j (x^{k-j} + \dots)$$

4: Apply with $j = k-1$:

$$|f(\alpha)|^{2^{k-1}} \ll P^{2^{k-1}-k} \sum_{|h_1| < P} \dots \sum_{|h_{k-1}| < P} \Upsilon(\underline{h})$$

where

$$\Upsilon(\underline{h}) = \sum_{x \in I_{k-1}(\underline{h})} e(\alpha \Delta_{k-1}(x^k; \underline{h}))$$

$$\leq \left| \sum_{x \in I_{k-1}(\underline{h})} e(k! h_1 \dots h_{k-1} x \alpha) \right|$$

Geometric progression
in x

$$\sum_{x < x \leq x+Y} e(\alpha x) = \frac{e(\alpha(x+Y+1)) - e(\alpha(x+1))}{e(\alpha) - 1}$$

$$\ll |\sin(\pi \alpha)|^{-1}$$

$$\ll \|\alpha\|^{-1} \quad (\alpha \neq 0)$$

$$\|\alpha\| = \min_{y \in \mathbb{Z}} |\alpha - y|$$

(trivial estimate is Y)

So

$$|f(\alpha)|^{2^{k-1}} \ll P^{2^{k-1}-k} \sum_{\substack{n \\ |h_i| < P}} \min \{P, \|k!h_1 \dots h_{k-1} \alpha\|^{-1}\}$$

$$\ll P^{2^{k-1}-k} \left(P^{k-1} + P^\varepsilon \sum_{1 \leq n \leq k!P^{k-1}} \min \{P, \|n\alpha\|^{-1}\} \right)$$

suppose $|\alpha - a/q| \leq q^{-2}$ & $(a, q) = 1$

$$\ll P^{2^{k-1} + \varepsilon} (q^{-1} + P^{-1} + qP^{-k})$$

$\Downarrow$

$$|f(\alpha)| \ll P^{1+\varepsilon} (q^{-1} + P^{-1} + qP^{-k})^{2^{1-k}}$$

(Weyl's inequality).

Now suppose $\alpha \in \mathbb{m}$

i.e. whenever $|\alpha - a/q| \leq \psi(P) P^{-k}$

and $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a, q) = 1$

then $q > \psi(P)$.

Use Dirichlet's theorem

$\exists a \in \mathbb{Z}$, $q \in \mathbb{N}$ with $(a, q) = 1$ and

$$|\alpha - a/q| \leq q^{-1} (\psi(P) P^{-k})$$

with $0 \leq a \leq q \leq \psi(P)^{-1} P^k$

$\Downarrow$
if $\alpha \in \mathbb{m}$, have $q > \psi(P)$

$$\Downarrow$$
$$f(\alpha) \ll P^{1+\varepsilon} (q^{-1} + P^{-1} + qP^{-k})^{2^{1-k}}$$

$$\ll P^{1+\varepsilon} (\psi(P)^{-1} + P^{-1} + (\psi(P)^{-1} P^k) P^{-k})^{2^{1-k}}$$

$$\ll P^{1+\varepsilon} \psi(P)^{-2^{1-k}}.$$

Potentially as strong as

$$f(\alpha) \ll P^{1-2^{1-k}+\varepsilon}$$

but in any case need $\psi(P) > P^\delta$ ($\delta > 0$)
to say something non-trivial.

Example: Consider

$$F(\underline{x}) = b_0 x_0^d + \dots + b_s x_s^d$$

with $s \geq d 2^{d-1}$, and investigate solutions
of $F(\underline{x}) = 0$ with $|x_i| \leq B$

$$g(\alpha) = \sum_{|x| \leq B} e(\alpha x^d), \quad g_i(\alpha) = g(b_i \alpha).$$

Minor arcs (assume $\psi(B) = B$):

$$\int_m |g_0(\alpha) \dots g_s(\alpha)| d\alpha \stackrel{\text{Hölder}}{\leq} \prod_{i=0}^s \left(\int_m |g_i(\alpha)|^{s+1} d\alpha \right)^{\frac{1}{s+1}}$$

$$\leq \max_{0 \leq i \leq s} \left(\sup_{\alpha \in m} |g_i(\alpha)| \right)^{s+1}$$

$$\ll \left(B^{1-2^{1-d}+\varepsilon} \right)^{s+1} = o\left(B^{s+1-d} \right) \quad s \geq d 2^{d-1}.$$

18.
Given non-degenerate binary form
 $\Phi(x, y) \in \mathbb{Z}[x, y]$ of degree d , can
 obtain

$$\sum_{|x|, |y| \leq B} e(\alpha \Phi(x, y)) \ll B^{2+\varepsilon} (q^{-1} + B^{-1} + qB^{-d})^{2^{-1}}$$

$$|\alpha - a/q| \leq q^{-2}, \quad (a, q) = 1 \quad (W' 1999)$$

Somewhat weaker in considering

$$\left| \sum_{\substack{x_0, \dots, x_s \\ |x_i| \leq B}} e(\alpha F(x_0, \dots, x_s)) \right|^{2^{d-1}} \quad (\text{Birch, 1961})$$

Need to consider small values mod 1
 assumed by collection of $s+1$ linear
 forms with varying coefficients

(analogous to $\sum_{1 \leq n \leq B^d} \min \{B, \| \alpha n \|^{-1}\}$).

Geometry of numbers — singular locus plays rol

XI. MINOR ARC ESTIMATES

- MEAN VALUES.

If diophantine problem has separable structure, can apply Hölder + mean values:

$$h(\alpha) = \sum_{x \in A} e(\alpha x^k)$$

$$b_0 x_0^k + \dots + b_s x_s^k = 0$$



$$\int_m |h(\alpha)|^{s+1} d\alpha \quad \text{important.}$$

$$\sup_{\alpha \in m} |h(\alpha)| \quad \wedge \quad \int_0^1 |h(\alpha)|^s d\alpha$$

But if $s = 2t$ is even,

$$\int_0^1 |h(\alpha)|^{2t} d\alpha = \int_0^1 h(\alpha)^t h(-\alpha)^t d\alpha$$

$$= \# \{ x_1^k + \dots + x_t^k = y_1^k + \dots + y_t^k; x_i, y_i \in A \}$$

Variant of Weyl's approach (Hua, 1938)

$$A = [1, P] : \int_0^1 |h(\alpha)|^{2^k} d\alpha \ll P^{2^k - k + \varepsilon}$$

Indiscreet with $A = \{n \in [1, P] \cap \mathbb{Z} : p|n \Rightarrow p \leq P^\eta\}$
 $\eta > 0$ small.

$$\int_0^1 |h(\alpha)|^{2^t} d\alpha \ll_{\varepsilon, \eta} P^{\lambda_t + \varepsilon}$$

$$\lambda_t = 2t - k, \quad t \geq k(\log k + \log \log k + O(1))$$

$$\lambda_t = 2t - k + ke^{1-2t/k} \quad (t \geq 2)$$

$$\lambda_t = t + \delta_t, \quad \delta_t < 4k^{1/2} e^{-\frac{16k}{e^{1/2}t}}$$

very close to t

Note:

$$\int_0^1 |h(\alpha)|^{2^t} d\alpha = \# \{x_1^k + \dots + x_t^k = y_1^k + \dots + y_t^k; x_i, y_i \in A\}$$

$$\gg P^t \quad (x_i = y_i)$$

"square-root" cancellation?

Work of Browning, Heath-Brown, Salberger

$$\int_0^1 \left| \sum_{1 \leq x \leq P} e(\alpha x^k) \right|^6 d\alpha \sim 3! P^3$$

for $k \geq 20$?

These mean-value estimates can be converted to Weyl estimates via large sieve:

$$\sup_{\alpha \in M} |h(\alpha)| \ll P^{1 - \frac{1}{(1+o(1))k \log k}}$$

Conjecture:

$$\int_0^1 \left| \sum_{1 \leq x \leq P} e(\alpha x^k) \right|^{2t} d\alpha \ll P^{t+\varepsilon}$$

$1 \leq t \leq k$

asymptotic formula (HP & WA) for

$$b_0 x_0^k + \dots + b_s x_s^k = 0$$

for $s \geq 2k$.

MYTH # 2.

Circle method only works if number of variables is large ($2 \times \Sigma \text{degrees}$).

Idea: Parameterise solution space and work inside this space

Example: One can reformulate

Gower's / Green-Tao as a pure application of circle method.

$$\sum_{j=0}^s a_{ij} x_j = b_i \quad (1 \leq i \leq r)$$

$$(s \geq r+1)$$

Solution set can be parametrised in $s+1-r$ (homogeneous) parameters - do Fourier analysis in this set (affine)

Limited by $\sqrt{\quad}$ -cancellation in this set.

Note: One can incorporate this idea
with descent / torsors

(see E. Peyre)

cf. Heath-Brown, Skorobogatov ...

$$t^a (1-t)^b = \alpha N_{K/\mathbb{Q}}(\underline{x})$$



$$A N_{K/\mathbb{Q}}(\underline{y}) + B N_{K/\mathbb{Q}}(\underline{z}) = C z^d$$

$$\# \left\{ N_{K/\mathbb{Q}}(\underline{x}) = n \neq 0 : \begin{array}{l} |\underline{x}| \leq B \end{array} \right\} \\ \ll (nB)^\varepsilon.$$

$$\int_0^1 \left| \sum_{|\underline{x}| \leq B} e(\alpha N_{K/\mathbb{Q}}(\underline{x})) \right|^2 d\alpha \\ \ll B^{d+\varepsilon}$$

$$[K:\mathbb{Q}] = d.$$