

# THE CIRCLE METHOD

A.K.A. THE HARDY-LITTLEWOOD  
METHOD.

## I. WHAT IS THE CIRCLE METHOD?

"ARITHMETIC HARMONIC ANALYSIS"?

A collection of strategies for estimating arithmetic coefficients (expected to be related to a product of local densities) in terms of behaviour of Fourier coefficients at rational points of small height.

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Best understood in terms of examples

- concentrate here on counting rational points on varieties
- also applicable to inequalities, more exotic arithmetic coefficients.

Typical Example: (of a problem)

Consider homogeneous polynomials  $F_i(\underline{x}) \in \mathbb{Q}[x_0, \dots, x_g]$  ( $1 \leq i \leq r$ ) that define a proper complete intersection

$$F_i(\underline{x}) = 0 \quad (1 \leq i \leq r). \quad (*)$$

When  $K$  is a field into which  $\mathbb{Q}$  embeds, consider

$$X(K) := \{ \underline{x} \in \mathbb{P}^g(K) : (*) \text{ is satisfied} \}$$

Problem 1: Prove Hasse Principle (HP) for (\*)  
i.e. Prove that whenever

$X(\mathbb{R}) \neq \emptyset$  and  $X(\mathbb{Q}_p) \neq \emptyset$  for all  $p$   
then

$$X(\mathbb{Q}) \neq \emptyset$$

Problem 2: Prove Weak Approximation (WA) for (\*)  
i.e. Prove that whenever  $\mathcal{P} \subset \{\text{primes}\} \cup \{\infty\}$  and

$\underline{a}_p \in X(\mathbb{Q}_p)$  for each  $p \in \mathcal{P}$ ,  
then for each  $\varepsilon > 0$ , there exists  $\underline{x} \in X(\mathbb{Q})$   
with  $|\underline{x} - \underline{a}_p| < \varepsilon$  for each  $p \in \mathcal{P}$ .

Observation: We can solve both HP & WA provided we are able to prove:

Given  $\underline{b} \in \mathbb{Z}^{s+1}$  and  $t \in \mathbb{N}$ , define

$$F_i(\underline{x}; t, \underline{b}) = F_i(t\underline{x} + \underline{b}) \quad (1 \leq i \leq r).$$

Then, given  $\mathcal{B} \subseteq (-1, 1)^s$ , one has

$$\begin{aligned} \# \{ \underline{x} \in \mathcal{B} \cap \mathbb{Z}^{s+1} : F_i(\underline{x}; t, \underline{b}) = 0 \quad (1 \leq i \leq r) \} \\ \sim \left( \prod_p v_p \right) B^{s+1 - \sum_1^r d_i} \quad \text{"product of local densities"} \end{aligned}$$

where  $d_i = \deg(F_i) \quad (1 \leq i \leq r)$ ,

$$\begin{aligned} v_p &= \text{"p-adic density of solutions"} \\ &= \lim_{h \rightarrow \infty} p^{h(r-s-1)} \# \{ \underline{x} \in (\mathbb{Z}/p^h\mathbb{Z})^{s+1} : F_i(\underline{x}; t, \underline{b}) \equiv 0 \pmod{p^h} \\ &\quad (1 \leq i \leq r) \} \end{aligned}$$

$v_\infty$  = surface area of real manifold defined by  $F_i(\underline{x}; t, \underline{b}) = 0 \quad (1 \leq i \leq r)$  in  $\mathcal{B}$ .

4. /  
Note 1: If we analyse situation for arbitrary (not necessarily homogeneous)  $F_i(x)$ , then this automatically takes care of  $F_i(x; t, b)$

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Note 2: We can only hope to achieve such an asymptotic formula when local-global principles hold (well-behaved).

- What if there is an obstruction to Hasse - Principle (Brauer - Manin, ...)?

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MYTH # 1.

The circle method is useful only when one expects HP (and WA) to hold.

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What to do if HP is suspected (feared, false)?

First use arithmetic / geometry to gather additional structure for variety, and hope to analyse solutions in terms of new object that is expected to satisfy HP & WA (e.g. by descent)

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Example: (Heath-Brown & Skorobogatov, Colliot-Thélène, Harari, Skorobogatov)

$$t^a (1-t)^b = \alpha N_{K/\mathbb{Q}}(x_1, \dots, x_n)$$

$$[\alpha \in \mathbb{Q}^*, a, b \in \mathbb{N}]$$

BM

$$A N_{K/\mathbb{Q}}(y_1, \dots, y_n) + B N_{K/\mathbb{Q}}(z_1, \dots, z_n) = C W^n$$

$$[A, B, C \in \mathbb{Z}^*]$$

HP  
&  
WA

## II. FOURIER SERIES.

Write

$$e(z) = e^{2\pi i z}$$

and make use of "fundamental theorem of Fourier analysis":

$$\int_0^1 e(\alpha h) d\alpha = \begin{cases} 0, & \text{when } h \in \mathbb{Z} \setminus \{0\} \\ 1, & \text{when } h = 0 \end{cases}$$

Given  $F(\underline{x}) \in \mathbb{Z}[x_0, \dots, x_s]$  and  $\mathcal{B} \subseteq (-1, 1)^{s+1}$ .

Then

$$\# \{ \underline{x} \in \mathcal{B} \cap \mathbb{Z}^{s+1} : F(\underline{x}) = 0 \}$$

$$= \sum_{\underline{x} \in \mathcal{B} \cap \mathbb{Z}^{s+1}} \int_0^1 e(\alpha F(\underline{x})) d\alpha$$

$$= \int_0^1 G(\alpha) d\alpha$$

where

$$G(\alpha) = \sum_{\underline{x} \in \mathcal{B} \cap \mathbb{Z}^{s+1}} e(\alpha F(\underline{x}))$$

7/ Use  $r$  times :

$$\# \{ \underline{x} \in B\mathbb{B} \cap \mathbb{Z}^{s+1} : F_i(\underline{x}) = 0 \quad (1 \leq i \leq r) \}$$

$$= \int_{[0,1]^r} H(\underline{\alpha}) d\underline{\alpha}$$

where

$$H(\underline{\alpha}) = \sum_{\underline{x} \in B\mathbb{B} \cap \mathbb{Z}^{s+1}} e(\alpha_1 F_1(\underline{x}) + \dots + \alpha_r F_r(\underline{x})).$$

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Next question — how does this relate to the product of local densities?

### III. RATIONAL ARGUMENTS.

For simplicity, consider now  $F(\underline{x}) \in \mathbb{Z}[x_0, \dots, x_s]$  and

$$G(\alpha) = \sum_{1 \leq x_0 \leq B} \dots \sum_{1 \leq x_s \leq B} e(\alpha F(\underline{x})).$$

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Consider  $\alpha = a/q$  with  $a \in \mathbb{Z}, q \in \mathbb{N}, (a, q) = 1$ .

$$G\left(\frac{a}{q}\right) = \sum_{b_0=1}^q \dots \sum_{b_s=1}^q \sum_{\frac{1-b_0}{q} \leq y_0 \leq \frac{B-b_0}{q}} \dots \sum_{\frac{1-b_s}{q} \leq y_s \leq \frac{B-b_s}{q}} e\left(\frac{a}{q} F(qy + \underline{b})\right)$$

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$$e\left(\frac{a}{q} F(qy + \underline{b})\right) = e\left(\frac{a}{q} F(\underline{b})\right)$$

↓

$$G\left(\frac{a}{q}\right) = \left(\frac{B}{q}\right)^{s+1} S(q, \underline{b}) + O\left(B^s q\right)$$

where

$$S(q, a) = \sum_{\underline{b} \in (\mathbb{Z}/q\mathbb{Z})^{s+1}} e\left(\frac{a}{q} F(\underline{b})\right)$$

$$G(\alpha) = \sum_{\underline{x} \in [1, B]^{s+1}} e(\alpha F(\underline{x}))$$

$$S(q, a) = \sum_{\underline{b} \pmod{q}} e\left(\frac{a}{q} F(\underline{b})\right)$$

Have shown

$$G\left(\frac{a}{q}\right) = q^{-(s+1)} S(q, a) B^{s+1} + O(B^s q)$$

Can use this to obtain asymptotic estimate for  $G(\alpha)$  when  $\alpha$  is "close" to  $\frac{a}{q}$ ,

$$\text{say } \alpha = \beta + \frac{a}{q}$$

Note that when  $F(\underline{x})$  has degree  $d$  and  $\underline{x} \in [1, B]^{s+1}$ , then

$$\left| \frac{\partial}{\partial x_i} e(\beta F(\underline{x})) \right| = O(|\beta| B^d)$$

$\Downarrow$

$$e(\beta F(\underline{x})) = 1 + O(|\beta| B^d)$$

Mean - Value Theorem:

$$G\left(\frac{a}{q} + \beta\right) = \sum_{\substack{\underline{b} \bmod q}} \sum_{\substack{\underline{y} \\ q\underline{y} + \underline{b} \in [1, B]^{s+1}}} e\left(\frac{a}{q} F(\underline{b})\right) e(\beta F(q\underline{y} + \underline{b}))$$

$$= q^{-(s+1)} S(q, a) v(\beta) + O(B^s q (1 + |\beta| B^d))$$

where

$$v(\beta) = \int_{[1, B]^{s+1}} e(\beta F(\underline{x})) d\underline{x}$$

Let us suppose that  $\psi(B) \nearrow \infty$   
as slowly as we choose.

Then when  $q \leq \psi(B)$  and  $|\beta| \leq \psi(B) B^{-d}$   
have asymptotic formula

$$G\left(\frac{a}{q} + \beta\right) = q^{-(s+1)} S(q, a) v(\beta) + O(B^s \psi(B)^2)$$

$\downarrow$   
 $B^{s+1}$

#### IV. MAJOR ARCS.

We have an asymptotic formula for  $G(\alpha)$  when  $\alpha$  is close to  $\frac{a}{q}$  with  $q$  small.

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Define

$$\mathcal{M}(q, a) = \left\{ \alpha \in [0, 1) : \left| \alpha - \frac{a}{q} \right| \leq \psi(B) B^{-d} \right\}$$

$$\mathcal{M} = \bigcup_{\substack{0 \leq a \leq q \leq \psi(B) \\ (a, q) = 1}} \mathcal{M}(q, a)$$

"major arcs".

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Then

$$\int_{\mathcal{M}} G(\alpha) d\alpha = \sum_{1 \leq q \leq \psi(B)} \sum_{\substack{a=1 \\ (a, q)=1}}^q \int_{\frac{a}{q} - \psi(B) B^{-d}}^{\frac{a}{q} + \psi(B) B^{-d}} G(\alpha) d\alpha$$

with

$$G(\alpha) = q^{-(s+1)} S(q, a) V(\alpha - a/q) + O(\psi(B)^2 B^s)$$

12.7

So

$$\int_m G(\alpha) d\alpha = \sum_{1 \leq q \leq \psi(B)} \sum_{\substack{a=1 \\ (a,q)=1 \\ \frac{a}{q} + \psi(B)B^{-d}}}^q q^{-(s+1)} S(q, a) \\ \times \int_{\frac{a}{q} - \psi(B)B^{-d}}^{\frac{a}{q} + \psi(B)B^{-d}} v(\alpha - \frac{a}{q}) d\alpha$$

$$+ O(\psi(B)^2 B^s \sum_{1 \leq q \leq \psi(B)} q \cdot \psi(B) B^{-d}) \\ = G(\psi(B)) J(\psi(B)) + O(\psi(B)^5 B^{s-d})$$

where

$$G(Q) = \sum_{1 \leq q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q q^{-(s+1)} S(q, a)$$

truncated  
"singular series"

$$J(Q) = \int_{-QB^{-d}}^{QB^{-d}} v(\beta) d\beta$$

truncated  
"singular integral"

3. More generally (simple exercise):

$$H(\underline{\alpha}) = \sum_{\underline{x} \in B\mathbb{B} \cap \mathbb{Z}^{s+1}} e(\alpha_1 F_1(\underline{x}) + \dots + \alpha_r F_r(\underline{x}))$$

$d_i = \deg F_i$

$$\mathcal{M} := \bigcup_{\substack{0 \leq a_1, \dots, a_r \leq Q \leq \psi(B) \\ (q, a_1, \dots, a_r) = 1}} \mathcal{M}(q, \underline{a})$$

$$\mathcal{M}(q, \underline{a}) := \left\{ \underline{\alpha} \in [0, 1)^r : \left| \alpha_i - \frac{a_i}{q} \right| \leq \psi(B) B^{-d_i} \right. \\ \left. (1 \leq i \leq r) \right\}$$

$$\int_{\mathcal{M}} H(\underline{\alpha}) d\underline{\alpha} = \mathcal{O}(\psi(B)) J(\psi(B)) + O(\psi(B)^{r+4} B^{s - \sum d_i})$$

where

$$\mathcal{O}(Q) = \sum_{1 \leq q \leq Q} \sum_{\substack{\underline{a} \pmod{q} \\ (q, a_1, \dots, a_r) = 1}} Q^{-(s+1)} S(q, \underline{a})$$

$$J(Q) = \int_{-QB^{-d_1}}^{QB^{-d_1}} \dots \int_{-QB^{-d_r}}^{QB^{-d_r}} v(\underline{\beta}) d\underline{\beta}$$

$$S(q, \underline{a}) = \sum_{\underline{r} \bmod q} e \left( \frac{a_1}{q} F_1(\underline{r}) + \dots + \frac{a_r}{q} F_r(\underline{r}) \right)$$

$$v(\underline{\beta}) = \int_{\mathbb{B}} e(\beta_1 F_1(\underline{x}) + \dots + \beta_r F_r(\underline{x})) d\underline{x}$$

So far, these are formal asymptotic formulae — have not discussed convergence issues.

On the other hand — we've assumed nothing so far about  $\underline{F}(\underline{x})$ .

## V. Minor Arcs.

We want an asymptotic formula for

$$\int_0^1 G(\alpha) d\alpha \quad \text{in place of} \quad \int_{\mathcal{M}} G(\alpha) d\alpha$$

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Define

$$\mathcal{M} = [0, 1) \setminus \mathcal{M} \quad \text{"minor arcs"},$$

Note:  $\alpha \in \mathcal{M} \Rightarrow \alpha$  is never too close  
to a rational number  
with small denominator  
( $\alpha$  has "large height")



If  $a \in \mathbb{Z}$ ,  $q \in \mathbb{N}$  satisfy  $(a, q) = 1$

and  $\left| \alpha - \frac{a}{q} \right| \leq \psi(B) B^{-d}$

then  $q > \psi(B)$ .

Have

$$\# \{ \underline{x} \in B \cap \mathbb{Z}^{s+1} : F(\underline{x}) = 0 \}$$

$$= \int_0^1 G(\alpha) d\alpha$$

$$= \int_m G(\alpha) d\alpha + \int_m G(\alpha) d\alpha$$

$$= O(\psi(B)) J(\psi(B)) + o(B^{s+1-d})$$

$$+ \int_m G(\alpha) d\alpha.$$

Philosophy: When  $F(\underline{x})$  is reasonably well-behaved, argument  $\alpha F(\underline{x}) \pmod{1}$  is randomly behaved (enough) when  $\alpha \in m$  so that there is considerable cancellation

in sum

$$\sum_{\underline{x} \in B \cap \mathbb{Z}^{s+1}} e(\alpha F(\underline{x}))$$

17. Ideally, square-root cancellation:

$$|G(\alpha)| = o(B^{(s+1)/2 + \epsilon})$$

for  $\alpha$  in suitably defined minor arcs.

Say something about techniques / issues surrounding minor arc estimates next time

Sufficient for now to say that in wide set of circumstances, one can prove

$$\sup_{\alpha \in m} |G(\alpha)| = o(B^{s+1-d})$$

$\Downarrow$

$$\int_m |G(\alpha)| d\alpha = o(B^{s+1-d})$$

$\Downarrow$

$$\# \{ \underline{x} \in B\mathbb{B} \cap \mathbb{Z}^{s+1} : F(\underline{x}) = 0 \}$$

$$= O(\psi(B)) J(\psi(B)) + o(B^{s+1-d})$$

## VI. SINGULAR INTEGRAL.

Recall

$$J(Q) = \int_{-QB^{-d}}^{QB^{-d}} v(\beta) d\beta$$

where

$$v(\beta) = \int_{B\beta} e(\beta F(x)) dx$$

If  $F(x)$  is non-singular ( $B$  does not contain a singular point) then

$$v(\beta) \ll B^{s+1} (1 + B^d |\beta|)^{-1+\delta}$$

( $\delta > 0$ )

follows by integrating by parts ( $\delta = s$ ) and almost as strong estimates with singularities

19.

$$\int_{\mathbb{R} \setminus [-QB^{-d}, QB^{-d}]} v(\beta) d\beta \ll B^{s+1} \int_{QB^{-d}}^{\infty} (1 + B^d |\beta|)^{-1+\delta} d\beta$$

$$= o(B^{s+1-d}).$$

↓

$$J(Q) = J + o(B^{s+1-d})$$

where

$$J = \int_{-\infty}^{\infty} v(\beta) d\beta$$

$$= B^{s+1-d} \tilde{J}$$

where

by change of  
variable.

$$\tilde{J} = \int_{-\infty}^{\infty} \int_{\mathbb{B}} e(\beta F(\gamma)) d\gamma d\beta$$

When  $F(\gamma) = 0$  defines non-singular variety in  $\mathbb{B}$ , can use Fourier's integral formula

$$\tilde{J} = \text{volume of real manifold } F(\gamma)=0 \text{ in } \mathbb{B}.$$

## VII. SINGULAR SERIES.

Recall

$$G(Q) = \sum_{1 \leq q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q q^{-(s+1)} S(q, a)$$

where

$$S(q, a) = \sum_{r \bmod q} e\left(\frac{a}{q} F(r)\right).$$

One can check that  $S(q, a)$  is quasi-multiplicative and thus

$$S(q) := \sum_{\substack{a=1 \\ (a,q)=1}}^q q^{-(s+1)} S(q, a)$$

is multiplicative. i.e.

$$(q, q') = 1 \Rightarrow S(qq') = S(q)S(q')$$

Then it suffices to consider  $S(q)$  when  $q = p^h$  for primes  $p$ .

21/  
 $S(p)$  can be estimated by using estimates of Deligne, OK when  $F(\underline{x})=0$  has singular locus of small dimension

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$S(p^h)$  can be analysed by imitating use of Hensel's lemma within exponential sum.

Again OK provided singular locus not too large.

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$\Downarrow$

$$S(p^h) \ll p^{-h(1+\delta)} \quad (\delta > 0, \text{ say})$$

In fact suffices

$\Downarrow$

$$S(q) \ll q^{-1-\delta/2}$$

$$S(p^h) \ll \begin{cases} p^{-h/d}, & h \text{ large} \\ p^{-1-\delta}, & \text{else} \end{cases}$$

Then

$$G(Q) = \sum_{1 \leq q \leq Q} S(q) \quad \text{is absolutely convergent as } Q \rightarrow \infty$$

Put

$$G := \sum_{q=1}^{\infty} S(q).$$

$$\begin{aligned} \text{Then } G - G(Q) &= \sum_{q > Q} S(q) \ll \sum_{q > Q} q^{-1-\delta} \\ &\ll Q^{-\delta} \end{aligned}$$

and

$$G = \prod_{p \text{ prime}} v(p)$$

where

$$v(p) = \lim_{H \rightarrow \infty} p^{-H(s)} \sum_{h=0}^H S(p^h)$$

$$= \lim_{H \rightarrow \infty} p^{-HS} \# \left\{ x \in (\mathbb{Z}/p^H \mathbb{Z})^s : F(x) \equiv 0 \pmod{p^H} \right\}$$

Moreover,

$$v(p) - 1 = O(p^{-1-\delta})$$

13/ Now, if there exists a non-singular  $p$ -adic solution of  $F(\underline{x}) = 0$  for each  $p$ , then

# solutions of  $F(\underline{x}) \equiv 0 \pmod{p^H}$   
 is  $\geq p^{H(s-w)}$ , some fixed  $w$ ,  
 for large enough  $H$  (Hensel's Lemma)

$\Downarrow$

$$\underline{v(p) > 0 \quad (\text{all } p)}$$

Thus, given existence of (non-singular)  $p$ -adic solutions for all  $p$ , have

$$\underline{G = \prod_p v_p \gg \prod_{p > p_0} (1 - p^{-1-\delta/3}) \gg 1.}$$

$$\# \{ \underline{x} \in \mathbb{B} \cap \mathbb{Z}^{s+1} : F(\underline{x}) = 0 \}$$

$$\sim G \tilde{J} B^{s+1-d} + o(B^{s+1-d})$$

$$\gg B^{s+1-d} \quad (\text{modulo (non-smg) local sol.}).$$

## VIII. Attainable Conclusions.

Theorem (Birch, 1961)

Let  $F_i(\underline{x}) \in \mathbb{Z}[x_0, \dots, x_s]$  ( $1 \leq i \leq r$ ) each have degree  $d$ . Define

$$V_{\text{sing}} = \bigcup_{\underline{m} \in \mathbb{C}^r} \{ \text{singular locus of } F_i(\underline{x}) = m_i \ (1 \leq i \leq r) \}$$

Then provided that

$$s - \dim(V_{\text{sing}}) \geq r(r+1)(d-1)2^{d-1},$$

one has

$$\begin{aligned} & \# \{ \underline{x} \in B\mathbb{B} \cap \mathbb{Z}^{s+1} : F_i(\underline{x}) = 0 \ (1 \leq i \leq r) \} \\ & \stackrel{N(B)}{=} \left( v_\infty \prod_p v_p \right) B^{s+1-rd} + o(B^{s+1-rd}) \end{aligned}$$

Theorem (Schmidt, 1985 ; W., 2006)

Let  $d$  be odd in the above. Then there exists  $\gamma = \gamma(d, r) < \infty$  such that

$$N(B) \gg B^{s-\gamma} \quad (s > \gamma)$$

15/

Diagonal forms:  $F_i(\underline{x}) = \sum_{j=0}^s a_{ij} x_j^d$

$d=2$ :  $s \geq 4r$  (Cook, 1972)

$d=3$ :  $s \geq 6$  ( $r=1$ ) (R. Baker, 1989)

$s \geq 12$  ( $r=2$ ) (J. Brüdern & W., ?)

$s \geq 6r+2$  (J. Brüdern & W., 2001)

$d$  large HP & WA for

$s \geq r(d \log d + O(d \log \log d))$ .

(W., 1992).

(Coefficients in "general position").

## Conclusions so far:

Major arc contribution  $\sim$  "product of local densities"

$$\sim \prod_p (\prod_p v_p) B^{s+1-d}$$

provided one takes  $\psi(B) \nearrow \infty$  slowly enough

— variety must be non-singular enough

singular integral — robust

singular series — singular locus plays a role

Minor Arcs — this is where hard work is to be found.