

Potential density of rational points

Brendan Hassett
Rice University

Notation and terminology

(2)

K field $\left\{ \begin{array}{l} \text{number field} \quad * \\ \text{fin. gen. over } \mathbb{Q} \quad * \\ \mathbb{C}(B) \quad B \text{ curve} \end{array} \right.$

X/K (smooth projective) variety
defined by polynomials
in K

$X(K)$ K rational points

Defn Rational points of X/K
are potentially dense if there
exists a finite extension L/K
such that $X(L)$ is Zariski
dense in X

Examples

③

① $X/\mathbb{C} = \{[x_0, x_1, x_2] : x_0^2 + x_1^2 + x_2^2 = 0\} \subset \mathbb{P}^2$

$$X(\mathbb{C}) = \emptyset$$

$X(\mathbb{C}(i))$ is dense in X

② $X/\mathbb{C}$ elliptic curve

If there is a point of infinite order

$$P \in X(L) \quad NP \neq 0, \text{ for } N \neq 0$$

Then, the multiples of P are dense in X

($\exists$ finite $L/\mathbb{C}$ over which such a point exists)

③ X curve of genus $g \geq 2$

$\Rightarrow$ Faltings $X(L)$ finite and thus not dense

General properties for varieties defined over given K (4)

1) $X \dashrightarrow Y$ dominating

Potential density holds for $X \Rightarrow$
" " " " Y
(birationally property)

2) $X \rightarrow Y$ étale morphism
of projective varieties

Potential density for $Y \Rightarrow$
" " " X

[Key ingredient over number fields:
Hermite's Thm: $\exists$ finite number of extensions
 L/K with bounded degree + discriminant]

3) Potential density holds for

- Rational varieties
- Unirational varieties
- Abelian varieties [Simple case]

Conjectural properties

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Defn: X is of general type if

$$\dim \Gamma(X, \omega_X^{\otimes n}) \sim n \dim(X) \quad n \gg 1$$

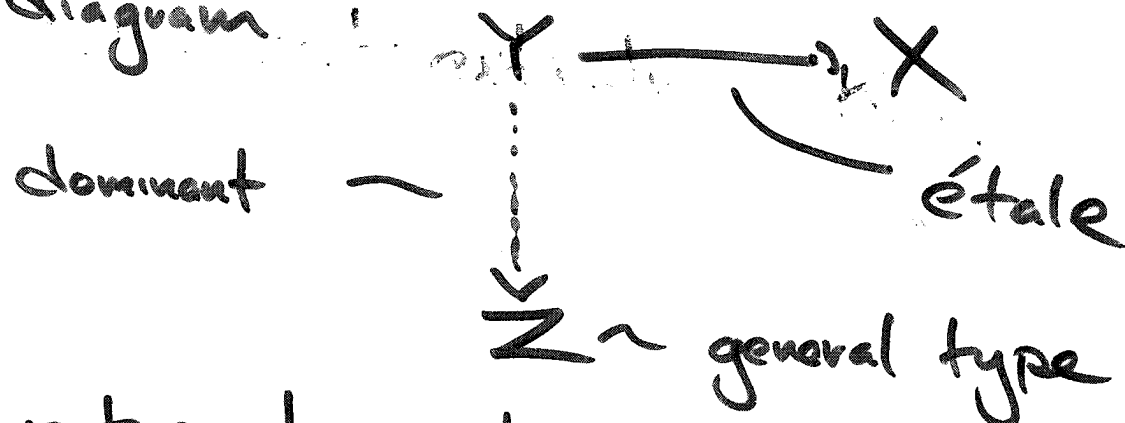
n -pluricanonical differentials

$(\dim(X) > 0)$

Conjecture (Lang-Bombieri) K number field
Rational points are not potentially
dense on varieties of general type

Panglossian converse: Suppose there is

no diagram



Then rational points on X are
potentially dense

Remark Campana has a more sophisticated
formulation $\leadsto$ special varieties

An observation over function fields (6)

Recall from yesterday's lecture:

$X \xrightarrow{\pi} B$ rationally connected
fibration over curve
 $b_1, \dots, b_r \in B$ so that X_{b_i} smooth
 $x_i \in X_{b_i}$

Then $\exists s: B \rightarrow X$ with $s(b_j) = x_j$
 $j=1, \dots, r$

Cor $K = \mathbb{C}(B)$ B curve

X/K rationally connected

$\Rightarrow X(K)$ is dense in X

[Even without extending K !]

Question X rationally connected / K number field
Does potential density hold?

Geometric conjecture: No examples of
non-unirational rationally connected
varieties are known!

Testing ground: Fano 3folds (7)

Ex. $X \subset \mathbb{P}^4$ quartic 3fold

while some quartic 3folds are unirational (Segre)

The general one is expected not to be unirational

Theorem (Harris-Tschinkel)

Rational points on quartic 3folds are potentially dense. More generally, any quartic hypersurface containing a line

$\ell \subset X \subset \mathbb{P}^n$ has this property

[More details to follow]

Summary: $/K$ finitely gen. over $\mathbb{Q}$ (8)

X smooth Fano 3fold

Then rational points are potentially dense except perhaps in the case where

X is a double cover of $\mathbb{P}^3$ branched over a sextic

$$X = \{v^2 = f_6(w, x, y, z)\}$$

Problem Resolve this last case completely

Remarks, 1) The case where

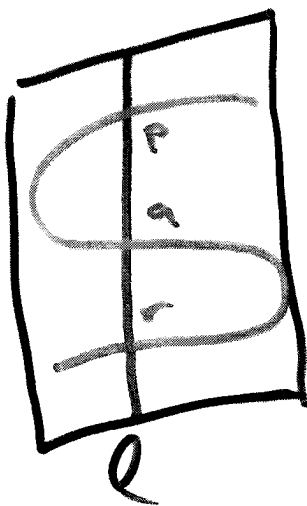
$\Sigma = \{f_6 = 0\}$ is singular has been studied by Cheltsov Park

2) Cases where Σ is smooth are known

Back to quartics containing a line (9)

$$l \subset X_4 \subset \mathbb{P}^n$$

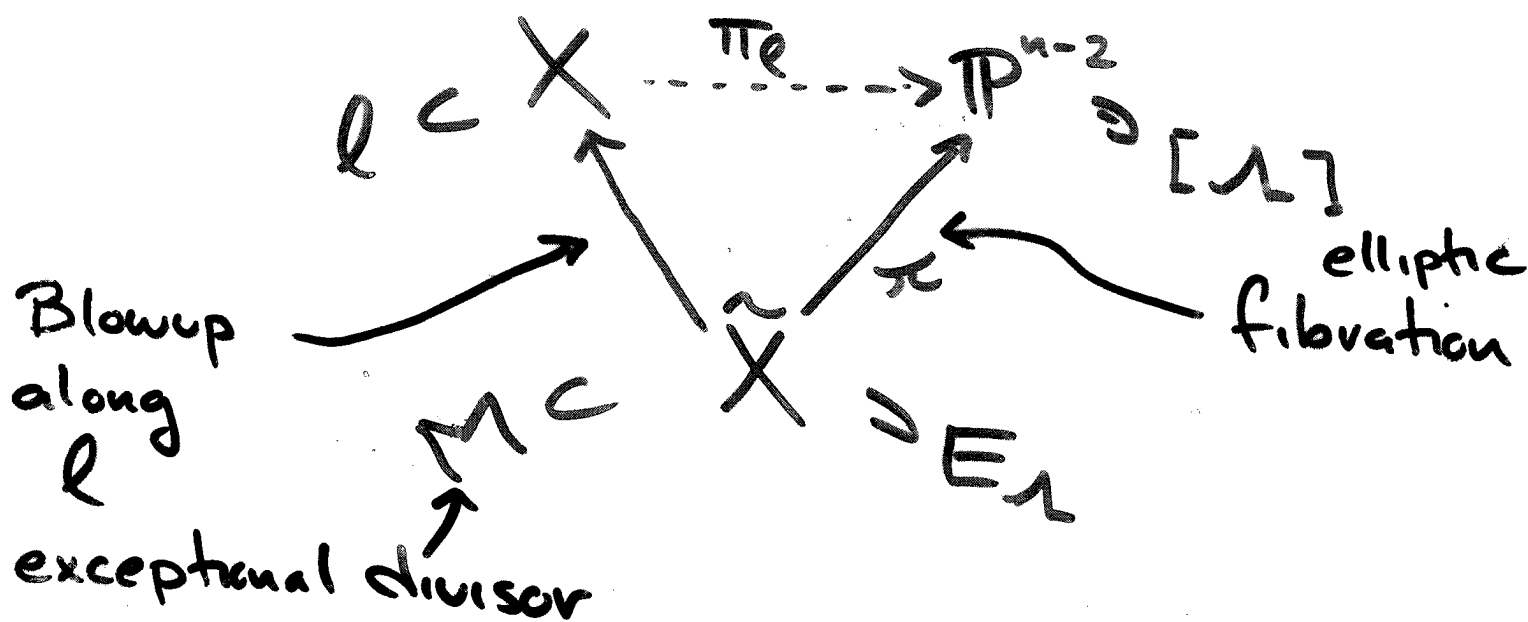
$$X_4 \cap \Lambda = l \cup E_\Lambda$$



E_Λ cubic plane curve

$\mathbb{P}^2 = \Lambda$ plane containing l

Projection from l gives an elliptic fibration



$$M \cap E_\Lambda = l \cap E_\Lambda = \{p, q, r\}$$

$\Rightarrow M$ is "trisection" of π

Step 1

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Basechange to M to get sections
(trivialize torsor)

$$Y = \hat{X} \times_{\mathbb{P}^{n-2}} M \longrightarrow M$$

1) Diagonal:

Yields rational
group law on
 Y over M

$$\begin{array}{ccc} \Delta_M: M & \longrightarrow & Y \\ \Delta_M \downarrow & & \parallel \\ M \times M & \subset & \hat{X} \times M \end{array}$$

2) Trace:

$$\tau_M: M \longrightarrow Y$$

But not
always!

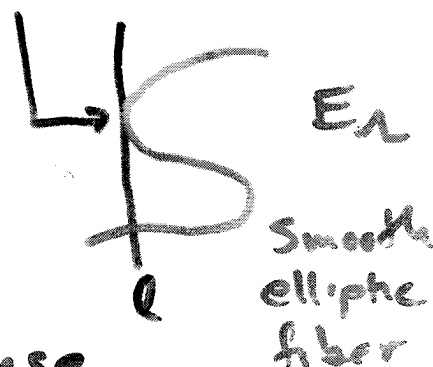
$$p \longmapsto \begin{array}{l} p+q+r-3p \\ q+r-2p \end{array}$$

Point where M ramifies

Step 2

Typically $\tau_M(M)$
has infinite order in the
group so

$$\bigcup_{n \in \mathbb{Z}} \frac{M[\tau_M(M)]}{M} \subset Y \text{ dense}$$



Step 3

M is a projective
bundle over $\mathbb{A}^1$, thus has dense
rational points, as does Y and X

Question

$$Q \subset X \subset \mathbb{P}^3$$

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smooth quartic surface, containing line
all defined over number field K

Is $X(K) \subset X$ dense?

Observation Key structures in argument

1) An elliptic fibration

$$\pi: \tilde{X} \longrightarrow \mathbb{P}^1$$

2) A multisection $M \subset \tilde{X}$

$[\pi: M \longrightarrow \mathbb{P}^1 \text{ generically finite surjective}]$

with

2)a) Potentially dense rational points

b) Infinite order trace

$$\tau_M: M \longrightarrow M \times_{\mathbb{P}^1} \tilde{X}$$

after basechange

X smooth
projective

$$g = h^0(\Omega_X^1) = 0$$

$$K_X = 0$$

$$K3$$

e.g. K3 surface
in $\mathbb{P}^3$

Sample Theorem (Bogomolov Tschinkel) (12)

X K3 surface with
elliptic fibration

$$X \longrightarrow \mathbb{P}^1$$

Then rational points are potentially
dense.

Original proof multisection $M \subset X$
is rational curve of large degree
[Nontorsion proof ingenious]

Second proof (H-, Kollár) M is

a genus one curve

[Nontorsion proof easy - but
 M is constructed via deformation
theory]

Question 1 $X \longrightarrow \mathbb{P}^1$ elliptic
surface w/o multiple fibers

Are rational points potentially
dense?

[There are counterexamples with multiple
fibers: Colliot-Thélène, Sw. Dyer, ~~Shankar~~
Skorobogatov]

Question 2 Are rational points
potentially dense for all K3
surfaces?

(This is consistent with our
"Panglossian" converse)

What is known over
number fields?

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X K3 surface

Rational points are potentially dense
if

- 1) X elliptic

- 2) $|\text{Aut}(X)| = \infty$ (Bogomolov
Technique!)

N.B. In all these cases, the Néron
Severi group

$$NS(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$$

has rank ≥ 2 . (And most K3
surfaces with $NS(X)$ of rank two are
covered by (1) or (2))

Problem Find a K3 surface X with
 $NS(X)$ of rank one (i.e. a "general" one)
with dense rational points

(van Luyk) Examples $/\mathbb{Q}$ with $NS(X) = \mathbb{Z}$
and $|X(\mathbb{Q})|$ infinite

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Best result for general K3 surfaces

Theorem (H. Teneinkel)

X K3 surface

Then for some $e > 0$ the
 e -fold symmetric product

$$\text{Sym}^e X = \underbrace{X \times \dots \times X}_{e \text{ times}} / \mathcal{G}_e$$

has potentially dense ratl pts.

Effective $X = X_{2g-2} \subset \mathbb{P}^2$

normally embedded K3

Then we can take $e = g$ or
even $e = 2$ provided g is square

Geometry Use abelian fibrations

$$\text{Sym}^e X \dashrightarrow \mathbb{P}^e$$

an active subject in holomorphic symplectic
geometry

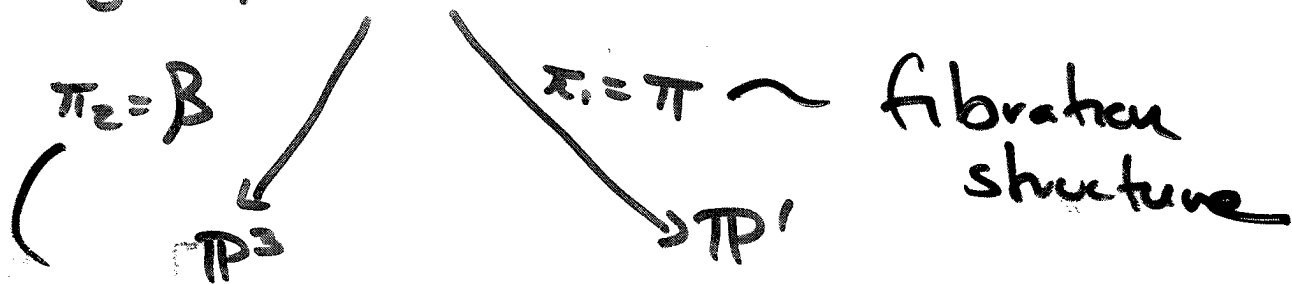
Potential density: K3 surfaces (16)
 over complex function fields
 $K = \mathbb{C}(B)$ B curve

Theorem (H-Tschinkel)

There exists a non-isotrivial K3 surface X over $\mathbb{C}(t)$ with $NS(X) = \mathbb{Z}$ and dense rational points

In fact: can take a general pencil of quartic surfaces in $\mathbb{P}^3$

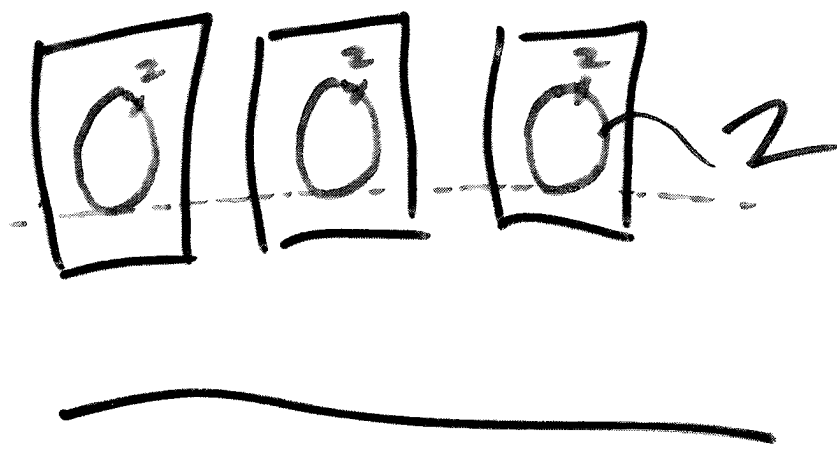
$$\{sf_4 + tg_4 = 0\} = \mathcal{X} \subset \mathbb{P}_{s,t}^1 \times \mathbb{P}^3$$



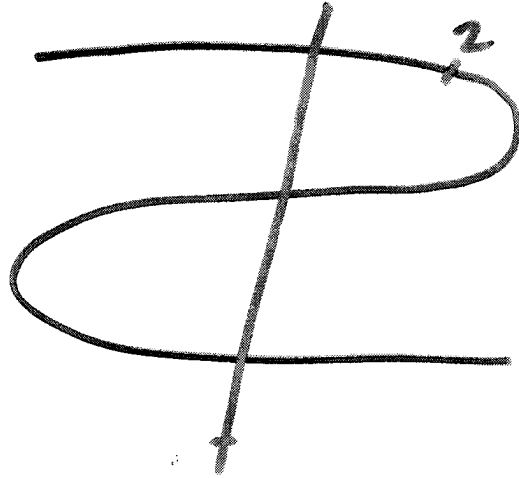
blow up along
 the curve
 $Z = \{f_4 = g_4 = 0\}$

Note: exceptional divisor
 is $\mathbb{P}^1 \times Z \subset \mathcal{X}$
 trivial family of curves
 inside $\mathcal{X}$

$$\begin{array}{c} \mathbb{P}^1 \times \mathbb{Z} \subset \mathbb{X} \\ \pi \downarrow \\ \mathbb{P}^1 \end{array}$$



$$\mathbb{Z} \subset \mathbb{P}^3$$



Heuristic:

(d=0) Each point $z \in \mathbb{Z}$ yields a section $\mathbb{P}^1 \times \mathbb{Z} \subset \mathbb{X}$ of π .

(d=1) Each 3-secant line to $\mathbb{Z}$ in $\mathbb{P}^3$ yields a section - its proper transform

Each $(4d-1)$ -secant rational curve of degree d yields a section of π

Parameter count suggest d -parameter family

Formal argument

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Lemma 1 $R \subset X_b$ smooth rational curve in smooth fiber

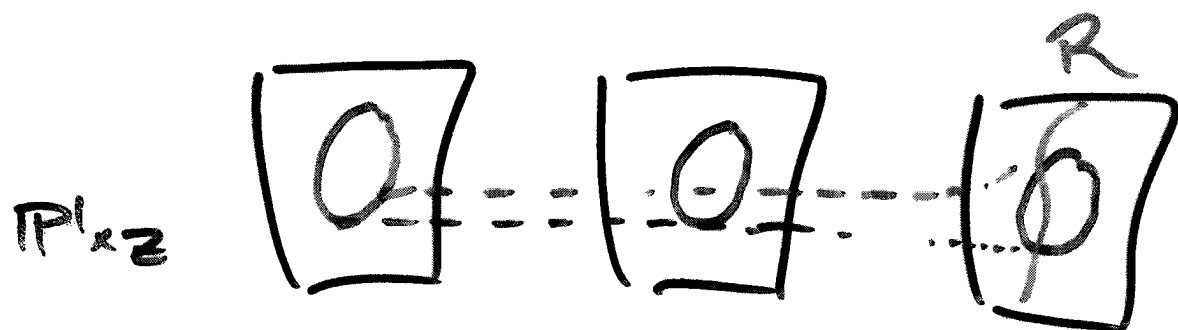
Assume $T(R/X) \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$

If $z \in Z \cap R$ Then

$$(\mathbb{P}^1 \times z) \cup_z R$$

deforms to a 1-parameter family of sections of X

The closure of those contains R



Section obtained by deforming

Lemma 2

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$$\mathbb{P}^{34} = \mathbb{P}(\text{Sym}^4 \mathbb{C}^4)$$

parameter space of
quadrics in $\mathbb{P}^3$

For each $d > 0$ there exists a
nonempty divisor

$$NL(d) \subset \mathbb{P}^{34}$$

with generic point parametrizing
a nonsingular quadric surface
with rational curve of
degree d

Lemma 3

Given a pencil $\mathcal{X}$
with classifying map $\downarrow$
 $\mathbb{P}^1$
 $\mu: \mathbb{P}^1 \longrightarrow \mathbb{P}^{34}$

A rational curve $R \in \mathcal{X}_b$ has
normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ if
 $\mathbb{P}^1$ is transverse to $NL(d)$ at b

So Take pencil with
classifying map
$$\mu: \mathbb{P}^1 \hookrightarrow \mathbb{P}^{3+1}$$

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transverse to $NL(d)$

for each $d > 0$ and so that

general fiber X_b has

$$NS(X_b) = \mathbb{Z} \cdot \text{hyperplane}$$

Problems 1) Are there examples
defined over number
fields, i.e. can the pencil
be written down over a
number field?

2) This construction applies for
K3 surfaces of degrees $2g-2$
 $2 \leq g \leq 9$

What about larger degrees?